

DENSITY PROBLEMS ON VECTOR BUNDLES AND MANIFOLDS

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ABSTRACT. We study some canonical differential operators on vector bundles over smooth, complete Riemannian manifolds. Under very general assumptions, we show that smooth, compactly supported sections are dense in the domains of these operators. Furthermore, we show that smooth, compactly supported functions are dense in second order Sobolev spaces on such manifolds under the sole additional assumption that the Ricci curvature is uniformly bounded from below.

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1. INTRODUCTION

In the analysis of differential operators, it is often useful in calculations to know that smooth, compactly supported functions are dense in the domain of the operator in question. We call this the *density problem*. In this paper, we show that the density problem can be solved in the positive for some canonical differential operators over a wide class of vector bundles.

More precisely, let \mathcal{V} be a smooth vector bundle over a smooth, complete Riemannian manifold \mathcal{M} . Suppose that \mathcal{V} is equipped with a metric h and connection ∇ that are compatible. Defining $\operatorname{div} = -\nabla^*$ in the L^2 theory, we show that $C_c^\infty(T^*\mathcal{M} \otimes \mathcal{V})$ is dense in $\mathcal{D}(\operatorname{div})$. Furthermore, letting $\Delta_B = -\operatorname{div} \nabla$, the *Bochner Laplacian* on \mathcal{V} , we show that $C_c^\infty(\mathcal{V})$ is dense in $\mathcal{D}(\Delta_B)$. In the case that $\mathcal{V} = \Omega(\mathcal{M})$, the exterior algebra over \mathcal{M} , we consider the operator d , the exterior derivative, and its adjoint $\delta = d^*$. In this situation, we show that $C_c^\infty(\Omega(\mathcal{M}))$ is dense in $\mathcal{D}(\delta)$.

While some of the results we present in this paper are known and can be accessed via alternative methods, our result on the density problem for second order Sobolev spaces on manifolds is new. The Sobolev space $W^{2,2}(\mathcal{M})$ is defined as the closure of functions $u \in C^\infty \cap L^2(\mathcal{M})$ satisfying $|\nabla u|, |\nabla^2 u| \in L^2(\mathcal{M})$ with respect to the

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norm $\|u\|_{W^{2,2}} = \|u\| + \|\nabla u\| + \|\nabla^2 u\|$, and $W_0^{2,2}(\mathcal{M})$ as the closure of $C_c^\infty(\mathcal{M})$ under the same norm. According to Hebey in [7], the best known conditions yielding $W_0^{2,2}(\mathcal{M}) = W^{2,2}(\mathcal{M})$ is to require both $\text{Ric} \geq \eta g$ and $\text{inj}(\mathcal{M}) \geq \kappa$, for some $\eta \in \mathbb{R}$ and $\kappa > 0$. We dispense the latter bound, yielding the following highlight theorem of this paper.

Theorem 1.1. *Let \mathcal{M} be a smooth, complete Riemannian manifold with metric g and Levi-Cevita connection ∇ . If there exists $\eta \in \mathbb{R}$ such that $\text{Ric} \geq \eta g$, then $W_0^{2,2}(\mathcal{M}) = W^{2,2}(\mathcal{M})$.*

Our motivation to study density problems emerges from the study of Kato square root type problems in the presence of geometry. The classical version of this problem on \mathbb{R}^n was solved by Auscher, Hofmann, Lacy, McIntosh and Tchamitchian in [1] and was rephrased in a first order point of view in [3] by Axelsson (Rosén), Keith and McIntosh. The proofs in the latter paper particularly exploit the density of compactly supported smooth functions and vector fields in the domains of the gradient and divergence operators respectively. Morris in [9] combines ideas from [3] and [2], rephrases and solves similar questions in the context of submanifolds in \mathbb{R}^n . There, density facts were needed but these were handled by different techniques. In formulating and solving a Kato square root problem on vector bundles by the author and McIntosh in [4], density problems became of central importance. While some of these issues were circumvented by alternative means, the desire to address density concerns persisted.

The main theme and philosophy in this paper is the following. Given a first order differential operator $D : C^\infty(\mathcal{V}) \rightarrow C^\infty(\mathcal{W})$ (where \mathcal{W} is another vector bundle), we construct an operator Π that is symmetric on $C_c^\infty(\mathcal{V}) \oplus C_c^\infty(\mathcal{W})$ such that it encodes D in a natural way. We then show that the density results follow from, or are sometimes equivalent to, showing that Π is essentially self-adjoint. The density problem for second order Sobolev spaces on manifolds follows from the essential self-adjointness of a particular Π^2 coupled with the lower bound on Ricci curvature. The paper [5] by Chernoff, introduced to the author by Baskin, justifies this reduction of density problems to essential self-adjointness, as it provides a set of very general conditions under which symmetric operators and their powers are essentially self-adjoint.

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2. PRELIMINARIES

2.1. Notation. Throughout this paper, we assume Einstein summation notation. Explicitly, whenever there is a raised and lowered index appearing multiplicatively, we assume summation over that index. For two quantities $a, b \geq 0$, we express inequalities up to a constant by writing $a \lesssim b$. By this we mean that there exists a $C > 0$ such that $a \leq Cb$. The constant C will be independent of the quantities a and b , and the dependence will be clear from context or preceding hypotheses. By writing $a \simeq b$, we mean that $a \lesssim b$ and $b \lesssim a$.

2.2. Operator theory. In this section, we provide an exposition of ideas from operator theory that we use in this paper. While some of the ideas here are valid for operators on general Banach spaces, we restrict ourselves to the theory in Hilbert spaces. We refer the reader to the excellent books [8] by Kato and [11] by Yosida which provide a more complete description of operator theory.

Let \mathcal{H}_1 and \mathcal{H}_2 be Hilbert spaces with inner products $\langle \cdot, \cdot \rangle_j : \mathcal{H}_j \times \mathcal{H}_j \rightarrow \mathbb{C}$. We say that a linear map $T : \mathcal{D}(T) \subset \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is an *operator* with *domain* $\mathcal{D}(T)$. If S is an operator such that $\mathcal{D}(S) \subset \mathcal{D}(T)$ and $Tu = Su$ for $u \in \mathcal{D}(S)$, then we write $S \subset T$ and say that T *extends* S . We emphasise that an operator is characterised by both the map and the domain.

An operator T is said to be *densely-defined* if $\overline{\mathcal{D}(T)} = \mathcal{H}_1$ and it is said to be *closed* if its graph, $\mathcal{G}(T) = \{(u, Tu) : u \in \mathcal{D}(T)\}$, is a closed subset of $\mathcal{H}_1 \times \mathcal{H}_2$. The latter notion is equivalent to requiring that whenever $u_n \in \mathcal{D}(T)$ such that $u_n \rightarrow u \in \mathcal{H}_1$ and $Tu_n \rightarrow v \in \mathcal{H}_2$, then $u \in \mathcal{D}(T)$ and $Tu = v$. Define the *operator norm* of T by $\|u\|_T = \|u\|_{\mathcal{H}_1} + \|Tu\|_{\mathcal{H}_2}$ whenever $u \in \mathcal{D}(T)$. Then, an operator T is closed if and only if $(\mathcal{D}(T), \|\cdot\|_T)$ is a Banach space. When we say “ X is dense in $\mathcal{D}(T)$,” we mean that $X \subset \mathcal{D}(T)$ is dense in the operator norm of T . We make a motivational remark that the notions closed and densely-defined are particularly useful when studying differential operators.

By the closed graph theorem (see Theorem 5.20 in [8]), a closed operator T with $\mathcal{D}(T) = \mathcal{H}$ is *bounded*, by which we mean there exists $C > 0$ such that $\|Tu\| \leq C\|u\|$ for $u \in \mathcal{H}$. Boundedness of an operator is equivalent to saying that it is continuous.

A notion that will be very important in later parts is that an operator T be *closable*. By this we mean that $\overline{\mathcal{G}(T)}$ is equal to the graph of another operator \bar{T} called the *closure* of T . An operator T is closable if and only if $u_n \in \mathcal{D}(T)$ with $u_n \rightarrow 0$ and $Tu_n \rightarrow v$ implies $v = 0$. It is immediate that $T \subset \bar{T}$.

We say that an operator $S : \mathcal{D}(S) \subset \mathcal{H}_2 \rightarrow \mathcal{H}_1$ is *adjoint* to T if $\langle Tu, v \rangle_{\mathcal{H}_2} = \langle u, Sv \rangle_{\mathcal{H}_1}$ for all $u \in \mathcal{D}(T)$ and $v \in \mathcal{D}(S)$. For such operators, if one is densely-defined, then the other is closable (see Theorem 5.28 in [8]). This is an important fact that we often use. For a given T , there are many operators S that are adjoint to T . However, if T is densely-defined, then there exists a unique maximal operator T^* adjoint to T called *the adjoint* of T . By maximal, we mean that if S is any operator adjoint to T , then $S \subset T^*$. We construct T^* in the following way. Let $v \in \mathcal{D}(T^*)$ if there exists $f \in \mathcal{H}_1$ such that $\langle Tu, v \rangle_{\mathcal{H}_2} = \langle u, f \rangle_{\mathcal{H}_1}$ for all $u \in \mathcal{D}(T)$ and define $T^*v = f$. The uniqueness of T^* follows since $\mathcal{D}(T)$ is dense in \mathcal{H} . If further T is closable, then T^* is densely-defined and closed, and furthermore, $T^{**} = \overline{T}$ (see Theorem 5.29 in [8]).

An important situation arises when $\mathcal{H} = \mathcal{H}_1 = \mathcal{H}_2$. There, we say an operator T is *hermitian symmetric* (or simply *symmetric*) to mean $\langle Tu, v \rangle = \langle u, Tv \rangle$, for all $u, v \in \mathcal{D}(T)$. The operator T is said to be *skew-symmetric* if $\langle Tu, v \rangle = \langle u, -Tv \rangle$. A symmetric operator is said to be *self-adjoint* if $T^* = T$. Explicitly, this means $Tu = T^*u$ for all $u \in \mathcal{D}(T) = \mathcal{D}(T^*)$. Note that a self-adjoint operator is necessarily densely-defined and closed. A densely-defined, closable operator T is said to be *essentially self-adjoint* if \overline{T} is self-adjoint. Self-adjointness will be one of the primary tools that we use in this paper. Although we run the risk of labouring the point, we present the following description of the maximal nature of self-adjointness.

Proposition 2.1. *Let T be self-adjoint, and S a symmetric extension of T . Then, $S = T$.*

Proof. First, since $T = S$ on $\mathcal{D}(T)$, it suffices to prove that $\mathcal{D}(S) \subset \mathcal{D}(T)$. By the fact that S is symmetric, we have that for all $u, v \in \mathcal{D}(S)$, $\langle Su, v \rangle = \langle u, Sv \rangle$. In particular, this reads $\langle Su, v \rangle = \langle u, Tv \rangle$ whenever $v \in \mathcal{D}(T)$. By the construction of the adjoint operator, we conclude that $u \in \mathcal{D}(T^*)$ and that $T^*u = Su$. But by the self-adjointness of T , $\mathcal{D}(T^*) = \mathcal{D}(T)$ and so we have that $\mathcal{D}(S) \subset \mathcal{D}(T)$. \square

The following proposition yields the same conclusion as above, but it is more useful since we only need to verify that the operators are equal on a dense subset.

Proposition 2.2. *Let T and S be self-adjoint. If $\mathcal{D} \subset \mathcal{D}(T) \cup \mathcal{D}(S)$, \mathcal{D} is dense in $\mathcal{D}(T)$, and $Tu = Su$ for all $u \in \mathcal{D}$, then $T = S$.*

Proof. Since S is self-adjoint, in particular it is symmetric, and thus, by Proposition 2.1, it suffices to show that $T \subset S$.

Let $u \in \mathcal{D}(T)$. By the hypotheses, there exists a sequence $u_j \in \mathcal{D}$ such that $u_j \rightarrow u$ and $Tu_j \rightarrow Tu$, and further $Tu_j = Su_j$. Thus, we have that $u_j \rightarrow u$ and $Su_j \rightarrow Tu$. But the self-adjointness of S in particular means that S is closed and hence $u \in \mathcal{D}(S)$ and $Su = Tu$. This shows that $T \subset S$ as required. \square

2.3. Hyperbolic equations and essential self-adjointness. The work of Chernoff in [5] is central to proving the results we present in this paper. Thus, for the convenience of the reader, we recall some notions from this paper.

Our setting is the following. Let \mathcal{M} be a smooth, complete Riemannian manifold with smooth metric g and volume measure $d\mu$. Further, let \mathcal{V} denote a smooth, complex vector bundle of finite rank over \mathcal{M} with smooth metric h . Since \mathcal{V} is complex, we assume that h is hermitian. We denote the fibre of \mathcal{V} over $x \in \mathcal{M}$ by \mathcal{V}_x . When we consider real bundles, we always implicitly identify them under their complexification.

By $\Gamma(\mathcal{V})$ denote the $d\mu$ -measurable *sections* of \mathcal{V} . That is, $\xi \in \Gamma(\mathcal{V})$ if $\xi : \mathcal{M} \rightarrow \mathcal{V}$, $\xi(x) \in \mathcal{V}_x$ and $d\mu$ -measurable in x . Then, we define $L^2(\mathcal{V})$ as the space of $\xi \in \Gamma(\mathcal{V})$ such that

$$\int_{\mathcal{M}} |\xi(x)|_x^2 d\mu(x) < \infty.$$

We note that $L^2(\mathcal{V})$ is a Hilbert space equipped with the inner product

$$\langle \xi, \eta \rangle = \int_{\mathcal{M}} h(\xi(x), \eta(x))_x d\mu(x).$$

Next, let L be a first order differential operator on $C^\infty(\mathcal{V})$, and let $L_c = L$ on $C_c^\infty(\mathcal{V})$. We recall the *symbol* of L from [5]. Let $x \in \mathcal{M}$, $v \in T_x^* \mathcal{M}$, $e \in \mathcal{V}_x$ and fix $g \in C^\infty(\mathcal{M})$ and $f \in C^\infty(\mathcal{V})$ such that $dg(x) = v$ and $f(x) = e$. Then, the symbol of L at x in the direction v acting on e is defined by

$$\sigma(x, v)e = \sigma_L(x, v)e = [L, gI]f(x).$$

We note that this is really the *principal* symbol of the operator L and emphasise as in [5] that this definition only applies to first order operators.

The *speed of propagation* at x is given by

$$c(x) = c_L(x) = \sup_{|v|=1} |\sigma(x, v)|$$

where $|\sigma(x, v)| = \sup_{|e|=1} |\sigma(x, v)e|$. Fix $x_0 \in \mathcal{M}$ and $r > 0$. Then, the speed of propagation of L inside the ball $B(x_0, r)$ by

$$c(r) = c_L(r) = \sup_{x \in B(x_0, r)} c(x).$$

With this notation at hand, we present the following theorem which is an immediate consequence of Theorem 2.2 in [5]. This result is the central tool that we use in this paper.

Theorem 2.3. *Let \mathcal{V} be a smooth vector bundle with smooth metric h over a smooth, complete Riemannian manifold \mathcal{M} . Let Π be a first order differential operator on $C^\infty(\mathcal{V})$ such that $\Pi_c = \Pi$ with domain $C_c^\infty(\mathcal{V})$ is symmetric. Furthermore, suppose that there exists $C > 0$ such that $c_\Pi(x) \leq C$ for each $x \in \mathcal{M}$. Then, every power of Π_c is essentially self-adjoint.*

Proof. We apply Theorem 2.2 in [5] which states that if L is a skew-symmetric operator on $C_c^\infty(\mathcal{V})$ and $\int_0^\infty c_L(r)^{-1} dr = +\infty$, then every power of $-iL$ is essentially self-adjoint.

First, let $L = i\Pi$. Then, it is easy to see that $L_c = L$ with domain $\mathcal{D}(L_c) = C_c^\infty(\mathcal{V})$ is skew-symmetric. Furthermore, an easy calculation yields that $\sigma_L(x, v)e = i\sigma_\Pi(x, v)e$. Therefore, $|\sigma_L(x, v)e| = |\sigma_\Pi(x, v)e|$ which implies that $c_L(r) = c_\Pi(r) \leq C$ after fixing a base point $x_0 \in \mathcal{M}$. Thus,

$$\int_0^\infty \frac{dr}{c_L(r)} = \int_0^\infty \frac{dr}{c_\Pi(r)} \geq \int_0^\infty \frac{dr}{C} = +\infty.$$

Putting these facts together demonstrates that L satisfies the hypotheses of Theorem 2.2 in [5] and therefore, we conclude that every power of the operator Π_c is essentially self-adjoint. \square

3. CONNECTIONS, DIVERGENCE AND LAPLACIANS ON VECTOR BUNDLES

In this section, we further assume that \mathcal{V} is equipped with a *connection* ∇ . Recall that this is a map $\nabla : C^\infty(\mathcal{V}) \rightarrow C^\infty(T^*\mathcal{M} \otimes \mathcal{V})$ satisfying the *Leibniz rule*

$$\nabla(fX) = \nabla f \otimes X + f\nabla X$$

for all $f \in C^\infty(\mathcal{M})$, $X \in C^\infty(\mathcal{V})$ and where $\nabla f = df$ is the exterior derivative on \mathcal{M} . Furthermore, let us assume that ∇ and h are *compatible*, by which we mean the following *product rule*

$$Xh(Y, Z) = h(\nabla_X Y, Z) + h(Y, \nabla_X Z)$$

holds for all $X \in C^\infty(T\mathcal{M})$, and $Y, Z \in C^\infty(\mathcal{V})$. By letting ∇_c be the restriction of ∇ to $C_c^\infty(\mathcal{V})$, we find that

$$(\star) \quad \langle \nabla_c u, v \rangle = \langle u, -\operatorname{tr} \nabla_c v \rangle,$$

for all $u \in C_c^\infty(\mathcal{V})$ and $v \in C_c^\infty(T^*\mathcal{M} \otimes \mathcal{V})$ (see Proposition 6.1 in [4] for a proof). This means the operators ∇_c and $-\operatorname{tr} \nabla_c$ are adjoint to each other and by the density of C_c^∞ in L^2 , both these operators are densely-defined and closable.

Define $-\operatorname{div} = \nabla_c^*$. Clearly, $\operatorname{tr} \nabla_c \subset \operatorname{div}$, and moreover, $\overline{\operatorname{tr} \nabla_c} \subset \operatorname{div}$. Furthermore, let $\tilde{\nabla} = (-\operatorname{tr} \nabla_c)^*$. It is clear that $\overline{\nabla_c} \subset \tilde{\nabla}$. In this section we show that under appropriate conditions, $\overline{\nabla_c} = \tilde{\nabla}$. The following proposition illustrates the connection of this statement to density problems.

Proposition 3.1. *Let \mathcal{M} be a smooth Riemannian manifold and \mathcal{V} a vector bundle with metric h and connection ∇ that are compatible. Then, the following are equivalent:*

- (i) $\overline{\operatorname{tr} \nabla_c} = \operatorname{div}$,
- (ii) $\overline{\nabla_c} = \tilde{\nabla}$,
- (iii) $C_c^\infty(\mathcal{V})$ is dense in $\mathcal{D}(\tilde{\nabla})$,
- (iv) $C_c^\infty(T^*\mathcal{M} \otimes \mathcal{V})$ is dense in $\mathcal{D}(\operatorname{div})$.

Proof. Suppose that $\overline{\text{tr } \nabla_c} = \text{div}$. Since $\overline{\nabla_c}$ is closed and densely-defined, $(-\text{div})^* = \nabla_c^{**} = \overline{\nabla_c}$. Also, $(-\text{div})^* = \overline{\text{tr } \nabla_c}^* = (\text{tr } \nabla_c)^* = \tilde{\nabla}$. Thus (i) implies (ii). By similar reasoning, (ii) implies (i). Now, we note that $\overline{\nabla_c} = \tilde{\nabla}$ if and only if $C_c^\infty(\mathcal{V})$ is dense in $\mathcal{D}(\tilde{\nabla})$ since $\nabla_c = \tilde{\nabla}$ on $C_c^\infty(\mathcal{V})$. Thus, (ii) and (iii) are equivalent. By similar argument, (iv) and (i) are equivalent to each other. This concludes the proof. \square

We also consider the following self-adjoint operators $\Delta_B = -\text{div } \overline{\nabla_c}$ with domain

$$\mathcal{D}(\Delta_B) = \{u \in L^2(\mathcal{V}) : u \in \mathcal{D}(\overline{\nabla_c}), \overline{\nabla_c}u \in \mathcal{D}(\text{div})\} \subset \mathcal{D}(\overline{\nabla_c})$$

and $\Delta_A = -\overline{\text{tr } \nabla_c} \tilde{\nabla}$ with domain

$$\mathcal{D}(\Delta_A) = \{u \in L^2(\mathcal{V}) : u \in \mathcal{D}(\tilde{\nabla}), \tilde{\nabla}u \in \mathcal{D}(\overline{\text{tr } \nabla_c})\} \subset \mathcal{D}(\tilde{\nabla}).$$

These are two *Laplacians* on the vector bundle. Furthermore, by (\star) , we find that the *connection Laplacian* $\Delta_c = -\text{tr } \nabla_c^2$ with domain $C_c^\infty(\mathcal{V})$ is densely-defined and closable. Let Δ_0 denote the closure of Δ_c and denote its domain by $\mathcal{D}(\Delta_0)$. Certainly, it is easy to see that $\Delta_0 \subset \Delta_B$ and $\Delta_0 \subset \Delta_A$ and we have the following proposition which lists some equivalences.

Proposition 3.2. *Under the same hypotheses as Proposition 3.1, the following are equivalent:*

- (i) Δ_0 is self-adjoint,
- (ii) $\Delta_0 = \Delta_B = \Delta_A$,
- (iii) $C_c^\infty(\mathcal{V})$ is dense in $\mathcal{D}(\Delta_B)$,
- (iv) $C_c^\infty(\mathcal{V})$ is dense in $\mathcal{D}(\Delta_A)$.

Proof. If Δ_0 is self-adjoint, then since $\Delta_0 \subset \Delta_B$ and $\Delta_0 \subset \Delta_A$ where Δ_B and Δ_A are in particular symmetric, Proposition 2.1 allows us to conclude that $\Delta_0 = \Delta_B$ and $\Delta_0 = \Delta_A$. Thus, (i) implies (ii). It is easy to see that (ii) implies (iii) since $\Delta_0 = \Delta_B$ on $C_c^\infty(\mathcal{V})$. Similarly, (ii) implies (iv). If (iv) holds, then $\Delta_0 = \Delta_A$ and Δ_A is self adjoint and so (iv) implies (i). Similarly, (iii) implies (i) and this concludes the proof. \square

We also define the following objects that we call *co-Laplacians*. First, define $\triangleright = -\overline{\nabla_c} \text{div}$ with domain

$$\mathcal{D}(\triangleright) = \{u \in L^2(T^*\mathcal{M} \otimes \mathcal{V}) : u \in \mathcal{D}(\text{div}), \text{div } u \in \mathcal{D}(\overline{\nabla_c})\} \subset \mathcal{D}(\text{div}).$$

Also, let $\tilde{\triangleright} = -\tilde{\nabla} \overline{\text{tr } \nabla_c}$ with domain

$$\mathcal{D}(\tilde{\triangleright}) = \{u \in L^2(T^*\mathcal{M} \otimes \mathcal{V}) : u \in \mathcal{D}(\overline{\text{tr } \nabla_c}), \overline{\text{tr } \nabla_c}u \in \mathcal{D}(\tilde{\nabla})\} \subset \mathcal{D}(\overline{\text{tr } \nabla_c}).$$

It is easy to see that both these operators are self-adjoint. The corresponding *connection co-Laplacian* is then given by \triangleright_c with domain $C_c^\infty(T^*\mathcal{M} \otimes \mathcal{V})$. As for the connection Laplacian, the condition (\star) implies that \triangleright_c is a densely-defined, closable operator. Thus, let \triangleright_0 denote the closure with domain $\mathcal{D}(\triangleright_0)$. It is easy to see that

$\triangleright_0 \subset \triangleright$ and $\triangleright_0 \subset \tilde{\triangleright}$. We have the following list of equivalences whose proof is similar to that of the analogous proposition we proved for the Laplacians.

Proposition 3.3. *Under the same hypotheses as Proposition 3.1, the following are equivalent:*

- (i) \triangleright_0 is self-adjoint,
- (ii) $\triangleright_0 = \triangleright = \tilde{\triangleright}$,
- (iii) $C_c^\infty(T^*\mathcal{M} \otimes \mathcal{V})$ is dense in $\mathcal{D}(\triangleright)$,
- (iv) $C_c^\infty(T^*\mathcal{M} \otimes \mathcal{V})$ is dense in $\mathcal{D}(\tilde{\triangleright})$.

We now prove the main result of this section. We remark that as a consequence of this theorem, each of the statements of Propositions 3.1, 3.2, and 3.3 hold.

Theorem 3.4. *Let \mathcal{M} be a smooth, complete Riemannian manifold with metric g and let \mathcal{V} be a smooth vector bundle over \mathcal{M} with smooth metric h and connection ∇ . Suppose that h and ∇ are compatible. Then, $\overline{\nabla_c} = \tilde{\nabla}$ and Δ_0 and \triangleright_0 are self-adjoint.*

Proof. Let $\mathcal{H} = L^2(\mathcal{V}) \oplus L^2(T^*\mathcal{M} \otimes \mathcal{V})$ and define

$$\Pi = \begin{pmatrix} 0 & -\text{tr } \nabla \\ \nabla & 0 \end{pmatrix}$$

with domain $\mathcal{D}(\Pi) = C^\infty(\mathcal{V}) \oplus C^\infty(T^*\mathcal{M} \otimes \mathcal{V})$. Let Π_c denote Π with domain $\mathcal{D}(\Pi_c) = C_c^\infty(\mathcal{V}) \oplus C_c^\infty(T^*\mathcal{M} \otimes \mathcal{V})$. First, we show that $\langle \Pi_c u, v \rangle = \langle u, \Pi_c v \rangle$, for $u, v \in \mathcal{D}(\Pi_c)$. Let $u = (u_1, u_2), v = (v_1, v_2) \in \mathcal{D}(\Pi_c)$. Then, $\Pi_c u = (-\text{tr } \nabla_c u_2, \nabla_c u_1)$ and $\Pi_c v = (-\text{tr } \nabla_c v_2, \nabla_c v_1)$. Therefore,

$$\begin{aligned} \langle \Pi_c u, v \rangle &= \langle -\text{tr } \nabla_c u_2, v_1 \rangle + \langle \nabla_c u_1, v_2 \rangle, \text{ and} \\ \langle u, \Pi_c v \rangle &= \langle u_1, -\text{tr } \nabla_c v_2 \rangle + \langle u_2, \nabla_c v_1 \rangle. \end{aligned}$$

But, by (\star) ,

$$\langle -\text{tr } \nabla_c u_2, v_1 \rangle = \langle u_2, \nabla_c v_1 \rangle \quad \text{and} \quad \langle \nabla_c u_1, v_2 \rangle = \langle u_1, -\text{tr } \nabla_c v_2 \rangle.$$

Therefore, $\langle \Pi_c u, v \rangle = \langle u, \Pi_c v \rangle$.

Next, we compute the symbol of Π . Fix $x \in \mathcal{M}$ and $v \in T_x^*\mathcal{M}$ and $e = (e_1, e_2) \in \mathcal{V}_x \oplus T_x^*\mathcal{M} \otimes \mathcal{V}_x$. Let $f \in C^\infty(\mathcal{V}) \oplus C^\infty(T^*\mathcal{M} \otimes \mathcal{V})$ such that $f(x) = e$, and $g \in C^\infty(\mathcal{M})$ such that $\nabla g = v$. Recall the following product rules for ∇ and $\text{tr } \nabla$:

$$\nabla(gf) = \nabla g \otimes f + g \nabla f, \quad \text{and} \quad \text{tr } \nabla(gf) = g \text{tr } \nabla v + \text{tr}(\overline{\nabla g} \otimes f)$$

where by $\overline{\nabla g}$ we mean complex conjugation. Write $f = (f_1, f_2)$. Then,

$$g\Pi f = g(-\text{tr } \nabla f_2, \nabla f_1)$$

and

$$\Pi(gf) = (-g \text{tr } \nabla f_2 - \text{tr}(\overline{\nabla g} \otimes f_2), \nabla g \otimes f_1 + g \nabla f_1).$$

Thus,

$$\begin{aligned}\sigma(x, v)e &= \Pi(gf)(x) - (g\Pi f)(x) \\ &= (-\operatorname{tr}(\overline{\nabla g(x)} \otimes f_2(x), \nabla g(x) \otimes f_1(x)) = (-\operatorname{tr}(\bar{v} \otimes e_2), v \otimes e_1)\end{aligned}$$

since $f(x) = (e_1, e_2)$. Therefore,

$$\begin{aligned}|\sigma(x, v)e|^2 &= |-\operatorname{tr}(\bar{v} \otimes e_2)|^2 + |v \otimes e_1|^2 \leq |v|^2 |e_2|^2 + |v|^2 |e_1|^2 \\ &= |v|^2 (|e_2|^2 + |e_1|^2) = |v|^2 |e|^2.\end{aligned}$$

From this, it is easy to see that $|\sigma(x, v)| = |v|$ making $c(x) = 1$.

These facts demonstrate that the hypotheses of Theorem 2.3 are satisfied for Π and so we conclude that Π_c is essentially self-adjoint. Letting Π_0 denote the closure of Π_c , note that

$$\Pi_0 = \begin{pmatrix} 0 & -\overline{\operatorname{tr} \nabla_c} \\ \overline{\nabla_c} & 0 \end{pmatrix} \quad \text{and} \quad \Pi_c^* = \begin{pmatrix} 0 & -\tilde{\nabla} \\ \tilde{\nabla} & 0 \end{pmatrix}.$$

By the self-adjointness of Π_0 , we find that $\Pi_c^* = \overline{\Pi_0}^* = \Pi_0^* = \Pi_0$. Thus, $\overline{\nabla_c} = \tilde{\nabla}$ and $\overline{\operatorname{tr} \nabla_c} = \operatorname{div}$.

Also, we have that

$$\Pi_c^2 = \begin{pmatrix} -\operatorname{tr} \nabla_c^2 & 0 \\ 0 & -\nabla_c \operatorname{tr} \nabla_c \end{pmatrix} = \begin{pmatrix} \Delta_c & 0 \\ 0 & \triangleright_c \end{pmatrix}.$$

By Theorem 2.3, Π_c^2 is essentially self-adjoint, which implies that $\overline{\Delta_c} = \Delta_0$ and $\overline{\triangleright_c} = \triangleright_0$ are also self-adjoint. \square

Remark 3.5. When \mathcal{V} is $\mathcal{M} \times \mathbb{C}$ (so that the sections of this bundle are measurable functions), the self-adjointness of Δ_0 on $L^2(\mathcal{M})$ under the assumption that \mathcal{M} is complete is a well known fact. It was stated in [5] as an application, and the author of this paper points to [6] by Gaffney and [10] by Roelcke as historical references.

Denote the (p, q) tensors on \mathcal{M} by $\mathcal{T}^{(p, q)}\mathcal{M}$ where p is the contravariant index and q the covariant index. We point out the following immediate consequence of this theorem for the tensor Laplacian and co-Laplacian.

Corollary 3.6. Let \mathcal{M} be a smooth, complete Riemannian manifold with metric g and connection ∇ that are compatible. Then, $C_c^\infty(\mathcal{T}^{(p, q)}\mathcal{M})$ is dense in $\mathcal{D}(\Delta_B)$ and $C_c^\infty(\mathcal{T}^{(p, q+1)}\mathcal{M})$ is dense in $\mathcal{D}(\triangleright)$.

Remark 3.7. We emphasise that in the corollary, we do not rule out the possibility that ∇ has torsion. That is, the relationship $\nabla_X Y - \nabla_Y X = [X, Y]$ (where $[\cdot, \cdot]$ denotes the Lie derivative) may fail in general for some $X, Y \in C^\infty(\operatorname{TM})$.

Next, we show some consequences of Theorem 3.4 to Sobolev spaces on vector bundles. Fix $k \in \mathbb{N}$ (although we only really deal with $k = 1, 2$). Let S_k be the set of $u \in C^\infty(\mathcal{V}) \cap L^2(\mathcal{V})$ such that $\nabla^i u \in C^\infty(\mathcal{V}) \cap L^2(\mathcal{T}^{(0, i)}\mathcal{M} \otimes \mathcal{V})$ for $1 \leq i \leq k$. Then, the Sobolev norm is defined as $\|u\|_{W_{k, 2}} = \|u\| + \sum_{i=1}^k \|\nabla^i u\|$ for $u \in S_k$. We define

the Sobolev spaces $W^{k,2}(\mathcal{V})$ as the completion of S_k under this norm. The Sobolev spaces $W_0^{k,2}(\mathcal{V})$ are defined as the completion of $C_c^\infty(\mathcal{V})$ under the same norm.

Let $\nabla_2 : S_1 \rightarrow C^\infty \cap L^2(T^*\mathcal{M} \otimes \mathcal{V})$ be defined by $\nabla_2 = \nabla$. Then, we have the following operator theoretic characterisations of $W^{1,2}(\mathcal{V})$ and $W_0^{1,2}(\mathcal{V})$.

Corollary 3.8. *Under the same hypotheses as Theorem 3.4, ∇_2 is a densely-defined, closeable operator and $W^{1,2}(\mathcal{V}) = \mathcal{D}(\overline{\nabla_2})$. Furthermore, $W_0^{1,2}(\mathcal{V}) = \mathcal{D}(\overline{\nabla_c})$ and $W_0^{1,2}(\mathcal{V}) = W^{1,2}(\mathcal{V})$.*

Proof. Fix $u \in S_1$ and $v \in C_c^\infty(T^*\mathcal{M} \otimes \mathcal{V})$. Since $x \mapsto h(u(x), -\text{tr } \nabla v(x))$ is compactly supported, an argument similar to that of the proof of Proposition 6.1 in [4] shows that $\langle \nabla_2 u, v \rangle = \langle u, -\text{tr } \nabla v \rangle$. This shows that ∇_2 is densely-defined and closeable. Furthermore, it shows that $\overline{\nabla_2} \subset \tilde{\nabla}$, and it is easy to see that $\overline{\nabla_c} \subset \overline{\nabla_2}$. Then by Theorem 3.4, we have that $\overline{\nabla_c} = \tilde{\nabla} = \overline{\nabla_2}$.

Since $\nabla_2 = \nabla$ on S_1 , it is easy to see that $W^{1,2}(\mathcal{V}) = \mathcal{D}(\overline{\nabla_2})$. We find that $W_0^{1,2}(\mathcal{V}) = \mathcal{D}(\overline{\nabla_c})$ for a similar reason, and this concludes the proof. \square

Remark 3.9. (i) *In this generality, do not know whether $W_0^{k,2}(\mathcal{V}) = W^{k,2}(\mathcal{V})$ for $k \geq 2$. In fact, even for the case $\mathcal{V} = \mathcal{M} \times \mathbb{C}$ with ∇ being the Levi-Cevita connection, we use assumptions on curvature to show that $W_0^{2,2}(\mathcal{M}) = W^{2,2}(\mathcal{M})$. We deal with this situation in §5. The case for $k > 2$ is considered in Proposition 3.2 in [7].*
(ii) *A more general version of this result is true. An alternative argument can be used to dispense the compatibility condition. See Proposition 2.3 in [4].*

Next, define $\Xi_k : S_k \rightarrow \bigoplus_{i=1}^k C^\infty \cap L^2(\mathcal{T}^{(0,i)}\mathcal{M} \otimes \mathcal{V})$ by $\Xi_k u = (\nabla^i u)_{i=1}^k$. The following proposition characterises Sobolev spaces in terms of this operator.

Proposition 3.10. *Under the same hypothesis as Theorem 3.4, the operator Ξ_k is closeable and $W^{k,2}(\mathcal{V}) = \mathcal{D}(\overline{\Xi_k})$.*

Proof. We note the case $k = 1$ is handled in Corollary 3.8.

Let $u_n \in S_k$ such that $u_n \rightarrow 0$ and $\Xi_k u_n \rightarrow (v_i)_{i=1}^k$. Then, we have that $v_1 = 0$ by Corollary 3.8. Set $w_n = \nabla u_n$ so that $w_n \rightarrow v_1 = 0$ and $\nabla w_n \rightarrow v_2$. Again, by invoking Corollary 3.8, we conclude that $v_2 = 0$. Proceeding this way, we find that $v_i = 0$ for all $i = 1, \dots, k$. Thus, Ξ_k is closeable.

That $W^{k,2}(\mathcal{V}) = \mathcal{D}(\overline{\Xi_k})$ follows from the fact that $\|u\|_{W^{k,2}} = \|u\|_{\Xi_k} = \|u\| + \|\Xi_k u\|$ for $u \in S_k$. \square

We conclude this section with the following regularity result that we use in §5 by specialising \mathcal{V} to $\mathcal{M} \times \mathbb{C}$.

Proposition 3.11. *Let \mathcal{M} be a smooth, complete Riemannian manifold and \mathcal{V} a smooth vector bundle with metric h and connection ∇ that are compatible. Then, $W_0^{2,2} \subset \mathcal{D}(\Delta_B)$ and $W^{2,2}(\mathcal{V}) \subset \mathcal{D}(\Delta_B)$ continuously.*

Proof. First, we show that whenever $u \in C^\infty \cap L^2(\mathcal{V})$ and $\text{tr } \nabla^2 u \in C^\infty \cap L^2(\mathcal{V})$, then $u \in \mathcal{D}(\Delta_B)$ and $\Delta_B u = -\text{tr } \nabla^2 u$. For that, fix $v \in C_c^\infty(\mathcal{V})$ and let $w \in C^\infty \cap L^2(T^*\mathcal{M} \otimes \mathcal{V})$ with $\text{tr } \nabla w \in C^\infty \cap L^2(\mathcal{V})$. Then, since $x \mapsto h(v(x), -\text{tr } \nabla w(x))$ is compactly supported, we have $\langle -\text{tr } \nabla w, v \rangle = \langle w, \nabla_c v \rangle$. Furthermore, by an application of the Cauchy-Schwarz inequality, we find that $\nabla u \in C^\infty \cap L^2(T^*\mathcal{M} \otimes \mathcal{V})$. Thus, upon putting $w = \nabla u$, we have that $\langle -\text{tr } \nabla^2 u, v \rangle = \langle \nabla u, \nabla_c v \rangle$. Again, since $x \mapsto g(\nabla u(x), \nabla_c v(x))_x$ is compactly supported, we have that $\langle \nabla u, \nabla_c v \rangle = \langle u, -\text{tr } \nabla_c^2 v \rangle$. In particular, for our choice of v , we have that $\Delta_B v = -\text{tr } \nabla_c^2 v$ and since Δ_B is self-adjoint, $u \in \mathcal{D}(\Delta_B)$ and $\Delta_B u = -\text{tr } \nabla^2 u$.

Now, we show that $W^{2,2}(\mathcal{V}) \subset \mathcal{D}(\Delta_B)$ continuously. So, fix $u \in W^{2,2}(\mathcal{V})$ and note that as a consequence of Proposition 3.10, there exists $u_n \in C^\infty \cap L^2(\mathcal{V})$ such that $u_n \rightarrow u$, and $\Xi_2 u_n \rightarrow \overline{\Xi_2} u$. In particular, $(\Xi_2 u_n)$ is Cauchy and therefore, $\|\nabla^2 u_n - \nabla^2 u_m\| \rightarrow 0$. Coupling this observation with the fact that $|\text{tr } \nabla^2 u_n| \leq |\nabla^2 u_n|$, and since $\Delta_B u_n = -\text{tr } \nabla^2 u_n$, we have that $\Delta_B u_n \rightarrow v$. But Δ_B is closed and hence, $u \in \mathcal{D}(\Delta_B)$ and $v = \Delta_B u$. The continuity of the embedding follows easily.

Since $W_0^{2,2}(\mathcal{V}) \subset W^{2,2}(\mathcal{V})$ is a continuous embedding, we conclude that $W_0^{2,2}(\mathcal{V}) \subset \mathcal{D}(\Delta_B)$ continuously. \square

4. EXTERIOR, INTERIOR DERIVATIVES AND THE DIRAC OPERATOR

In this section, we consider the density problem for some natural operators over the exterior algebra of \mathcal{M} . To fix notation, let the space of p -forms on \mathcal{M} be denoted by $\Omega^p(\mathcal{M})$. Let the fibre of the bundle $\Omega^p(\mathcal{M})$ over $x \in \mathcal{M}$ be denoted by $\Omega_x^p(\mathcal{M})$. Without causing too much confusion, let us denote the canonical extension of the metric g on \mathcal{M} to $\Omega^p(\mathcal{M})$ again by g . The *exterior product* of a p -form ω and q -form η is denoted by $\omega \wedge \eta \in \Omega^{p+q}(\mathcal{M})$, and the *interior* or *cut* product by $\omega \lrcorner \eta \in \Omega^{q-p}(\mathcal{M})$ which is defined as the dual to \wedge . The exterior algebra is then denoted by $\Omega(\mathcal{M}) = \bigoplus_{p=0}^n \Omega^p(\mathcal{M})$. This is a graded algebra with respect to \wedge . By $\Omega_x(\mathcal{M})$, we denote the fibre of $\Omega(\mathcal{M})$ over x .

The *exterior* and *interior* derivatives are then defined by the local expressions

$$d_\infty \omega = dx^i \wedge \nabla_{\partial_i} \omega \quad \text{and} \quad \delta_\infty \omega = -dx^i \lrcorner \nabla_{\partial_i} \omega$$

with domains $\mathcal{D}(d_\infty) = \mathcal{D}(\delta_\infty) = C^\infty(\Omega(\mathcal{M}))$. On p -forms, $d_\infty : C^\infty(\Omega^p(\mathcal{M})) \rightarrow C^\infty(\Omega^{p+1}(\mathcal{M}))$ and $\delta_\infty : C^\infty(\Omega^p(\mathcal{M})) \rightarrow C^\infty(\Omega^{p-1}(\mathcal{M}))$. By d_c and δ_c , denote the operators d_∞ and δ_∞ but with $\mathcal{D}(d_c) = \mathcal{D}(\delta_c) = C_c^\infty(\Omega(\mathcal{M}))$. Recall also that

$$\langle d_c \omega, \eta \rangle = \langle \omega, \delta_c \eta \rangle$$

for $\omega, \eta \in C_c^\infty(\Omega(\mathcal{M}))$. Thus, d_c and δ_c are densely-defined, closable operators. We denote the closures by d_0 and δ_0 respectively. Also, let $d = \delta_c^*$ and $\delta = d_c^*$. Then, we have the following theorem.

Theorem 4.1. *Let \mathcal{M} be a smooth, complete Riemannian manifold with smooth metric g and Levi-Cevita connection ∇ . Then, $d_0 = d$ and $\delta_0 = \delta$. In other words, $C_c^\infty(\Omega(\mathcal{M}))$ is dense in $\mathcal{D}(d)$ and in $\mathcal{D}(\delta)$. Furthermore, $C_c^\infty(\Omega(\mathcal{M}))$ is dense in $\mathcal{D}(d\delta)$ and in $\mathcal{D}(\delta d)$.*

Proof. Define

$$\Pi_\infty = \begin{pmatrix} 0 & \delta_\infty \\ d_\infty & 0 \end{pmatrix}$$

with $\mathcal{D}(\Pi_\infty) = C^\infty(\Omega(\mathcal{M})) \oplus C^\infty(\Omega(\mathcal{M}))$. Then, let Π_c to be the operator Π_∞ with $\mathcal{D}(\Pi) = C_c^\infty(\Omega(\mathcal{M})) \oplus C_c^\infty(\Omega(\mathcal{M}))$.

First, we show that Π_c is symmetric. Fix $u = (u_1, u_2), v = (v_1, v_2) \in C_c^\infty(\Omega(\mathcal{M}))$, and note that $\Pi_c u = (\delta_c u_2, d_c u_1)$, $\Pi_c v = (\delta_c v_2, d_c v_1)$. Then,

$$\langle \Pi_c u, v \rangle = \langle \delta_c u_2, v_1 \rangle + \langle d_c u_1, v_2 \rangle = \langle u_2, d_c v_1 \rangle + \langle u_1, \delta_c v_2 \rangle = \langle u, \Pi_c v \rangle.$$

Now, fix $v \in T^*\mathcal{M}$ and $e \in \Omega_x(\mathcal{M})$. Let $g \in C^\infty(\mathcal{M})$ and $f \in C^\infty(\Omega(\mathcal{M})) \oplus C^\infty(\Omega(\mathcal{M}))$ such that $dg(x) = v$ and $f = (f_1, f_2)$ and $f(x) = e = (e_1, e_2)$. Then,

$$\sigma(x, v)e = [\Pi_\infty, gI] f(x) = \Pi_\infty(gf) - g\Pi_\infty f = (\delta_\infty(gf_2) - g\delta_\infty f_2, d_\infty(gf_1) - gd_\infty f_1).$$

Note then that,

$$d_\infty(gf_1) - gd_\infty f_1 = (d_\infty g \wedge f_1 + gd_\infty f_1) - gd_\infty f_1 = d_\infty g \wedge f_1.$$

Thus, at x , $d_\infty g \wedge f_1 = v \wedge e_1$. By a similar calculation,

$$\delta_\infty(gf_2) - g\delta_\infty f_2 = (g\delta_\infty f_2 - d_\infty g \lrcorner f_2) - g\delta_\infty f_2 = -d_\infty g \lrcorner f_2$$

so that at x , $d_\infty g \lrcorner f_2 = v \lrcorner e_2$. We combine these two calculations so that

$$|\sigma(x, v)e|^2 = |v \wedge e_1|^2 + |v \lrcorner e_2|^2 \leq |v|^2 (|e_1|^2 + |e_2|^2) \leq |v|^2 |e|^2.$$

Therefore, $|\sigma(x, v)| = |v|$ and thus, $c(r) = 1$. Therefore, by Theorem 2.3, we conclude that Π_c and powers of Π_c are essentially self-adjoint. Thus, $\Pi_c^* = \overline{\Pi_c^*} = \overline{\Pi_c}$ and since

$$\Pi_c^* = \begin{pmatrix} 0 & \delta \\ d & 0 \end{pmatrix},$$

we conclude that $\overline{d_c} = d$ and $\overline{\delta_c} = \delta$.

Next, note that

$$\Pi_c^2 = \begin{pmatrix} \delta_c d_c & 0 \\ 0 & d_c \delta_c \end{pmatrix}$$

and by similar reasoning as above, we have that $\overline{\delta_c d_c}$ and $\overline{d_c \delta_c}$ are self-adjoint. It is easy to see that $\overline{\delta_c d_c} \subset \delta d$ and $\overline{d_c \delta_c} \subset d\delta$. Furthermore, δd and $d\delta$ are self-adjoint and so by Proposition 2.1, we conclude that $\overline{\delta_c d_c} = \delta d$ and $\overline{d_c \delta_c} = d\delta$. \square

Also, define $D_\infty : C^\infty(\Omega(\mathcal{M})) \rightarrow C^\infty(\Omega(\mathcal{M}))$ by $D_\infty = d_\infty + \delta_\infty$ and $D_c = d_c + \delta_c$ with domain $\mathcal{D}(D_c) = C_c^\infty(\Omega(\mathcal{M}))$. An easy calculation shows that

$$\langle D_c \omega, \eta \rangle = \langle \omega, D_c \eta \rangle$$

when $\omega, \eta \in C_c^\infty(\Omega(\mathcal{M}))$. Thus, D_c is a densely-defined, closable operator. Denote its closure by D_0 and let $D = d + \delta$. The latter operator is the *Hodge-Dirac* operator with domain $\mathcal{D}(D) = \mathcal{D}(d) \cap \mathcal{D}(\delta)$. As a consequence of Theorem 4.1, D is self-adjoint. The *Hodge-Laplacian* is then denoted by D^2 . We have the following theorem which is well known and written as an application in [5] but we include it here for completeness.

Theorem 4.2. *Let \mathcal{M} be a smooth, complete Riemannian manifold with smooth metric g endowed with the Levi-Cevita connection ∇ . Then, we have that $D_0 = D$ and $\overline{D_c^2} = D^2$. In other words, $C_c^\infty(\Omega(\mathcal{M}))$ is dense in $\mathcal{D}(D)$ and in $\mathcal{D}(D^2)$.*

Proof. It is straightforward to check that D_c is symmetric. The computation of the symbol was done in the proof of Theorem 4.1. Explicitly

$$\sigma(x, v)e = v \wedge e - v \lrcorner e,$$

and thus $c(r) = 1$. Thus, by Theorem 2.3, we have that D_0 is self-adjoint. But $D_0 \subset D$ and D is self-adjoint so by Proposition 2.1, we have that $D_0 = D$. Similarly, $\overline{D_c^2} = D^2$. \square

5. SECOND ORDER SOBOLEV SPACES ON MANIFOLDS

Throughout this section, in addition to the smoothness and completeness assumptions on \mathcal{M} and g , we further assume that \mathcal{M} is endowed with the Levi-Cevita connection ∇ . In this setting, we show that $W_0^{2,2}(\mathcal{M}) = W^{2,2}(\mathcal{M})$ whenever $\text{Ric} \geq \eta g$ for some $\eta \in \mathbb{R}$. The best currently known result according to Hebey in [7] requires the additional assumption that $\text{inj}(\mathcal{M}) \geq \kappa > 0$. This is the content of Proposition 3.3 in [7].

Let us recall some notation from §3, but specialised to the case that \mathcal{V} is $\mathcal{M} \times \mathbb{C}$. Let $\nabla_c = \nabla$ and $\Delta = -\text{tr} \nabla^2 : C^\infty(\mathcal{M}) \rightarrow C^\infty(\mathcal{M})$ denote the Laplacian on smooth functions. Furthermore, let $\Delta_c = \Delta$ with $\mathcal{D}(\Delta_c) = C_c^\infty(\mathcal{M})$. Since this is a densely-defined, closeable operator, let Δ_0 be its closure with domain $\mathcal{D}(\Delta_0)$. Recall that the Bochner Laplacian is the self-adjoint operator given by $\Delta_B = -\text{div} \overline{\nabla_c}$ with domain $\mathcal{D}(\Delta_B)$.

With this notation, we prove the highlight theorem of this paper.

Proof of Theorem 1.1. We note that the assumptions made in Theorem 1.1 satisfy the hypotheses of Theorem 3.4 and Proposition 3.11 with \mathcal{V} as $\mathcal{M} \times \mathbb{C}$. By the latter result, we have that $W_0^{2,2}(\mathcal{M}) \subset \mathcal{D}(\Delta_B)$ and $W^{2,2}(\mathcal{M}) \subset \mathcal{D}(\Delta_B)$ continuously.

Next, recall the following Bochner-Lichnerowicz-Weitzenböck identity

$$\langle \nabla \Delta u, \nabla u \rangle = \frac{1}{2} \Delta |\nabla u|^2 + |\nabla^2 u|^2 + \text{Ric}(\nabla u, \nabla u)$$

for $u \in C^\infty(\mathcal{M})$. By using the bound $\text{Ric} \geq \eta g$, we obtain that $\|\nabla^2 u\| \lesssim \|(I + \Delta)u\|$ whenever $u \in C_c^\infty(\mathcal{M})$. Thus, coupled with the fact that $\Delta_0 = \Delta_B$ by Theorem 3.4, we find the reverse inclusion $\mathcal{D}(\Delta_B) \subset W_0^{2,2}(\mathcal{M})$ holds and this allows us to conclude that $\mathcal{D}(\Delta_B) = W_0^{2,2}(\mathcal{M})$.

Combining these set inclusions, we have that $W^{2,2}(\mathcal{M}) \subset \mathcal{D}(\Delta_B) = W_0^{2,2}(\mathcal{M})$. But $W_0^{2,2}(\mathcal{M}) \subset W^{2,2}(\mathcal{M})$ continuously and so we conclude that $W_0^{2,2}(\mathcal{M}) = W^{2,2}(\mathcal{M})$. \square

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